A System of Interacting Neurons with Short Term Synaptic Facilitation

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Workshop on Mathematical Modeling and Statistical Analysis in Neuroscience

Outline

Contents

- This is a joint work with Antonio Galves and Errico Presutti.

1 Introduction: delayed responses, working memory, persistent activity and all that

It starts with Fuster in 1973

> A delayed-response trial typically consists of the presentation of one of two possible visual cues, an ensuing period of enforced delay and, at the end of it, a choice of motor response in accord with the cue. The temporal separation between cue and response is the principal element making the delayed response procedure a test of an operationally defined short-term memory function.

Reference: Fuster J. (1973) *Unit Activity in Prefrontal Cortex During Delayed-Response Performance: Neuronal Correlates of Transient Memory. J. Neurophys.* **36**: 61-78.

Fuster's paradigm

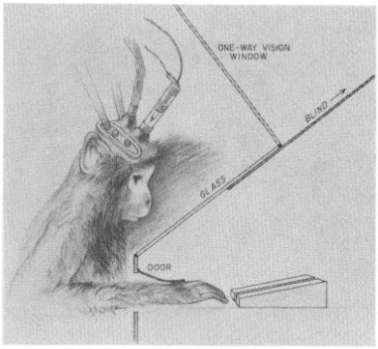


FIG. 1. Diagram of an experimental animal in the testing apparatus.

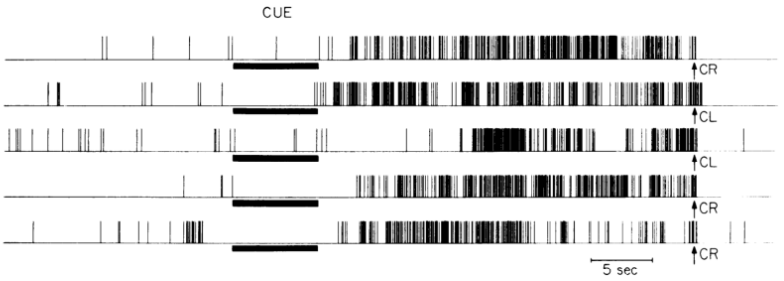


FIG. 4. Unit of type D during five delayed-response trials with 32-sec delay. Spikes are represented by vertical lines in a graphic display obtained by computer method. The notation next to the arrow at the end of each trial's delay refers to the accuracy (C, correct; I, incorrect) and side (R, right; L, left) of the response. The series of single-trial records in this figure—as in subsequent figures—is made of records from consecutive trials.

Figures 1 and 4 of Fuster (1973).

Other delayed activities are observed

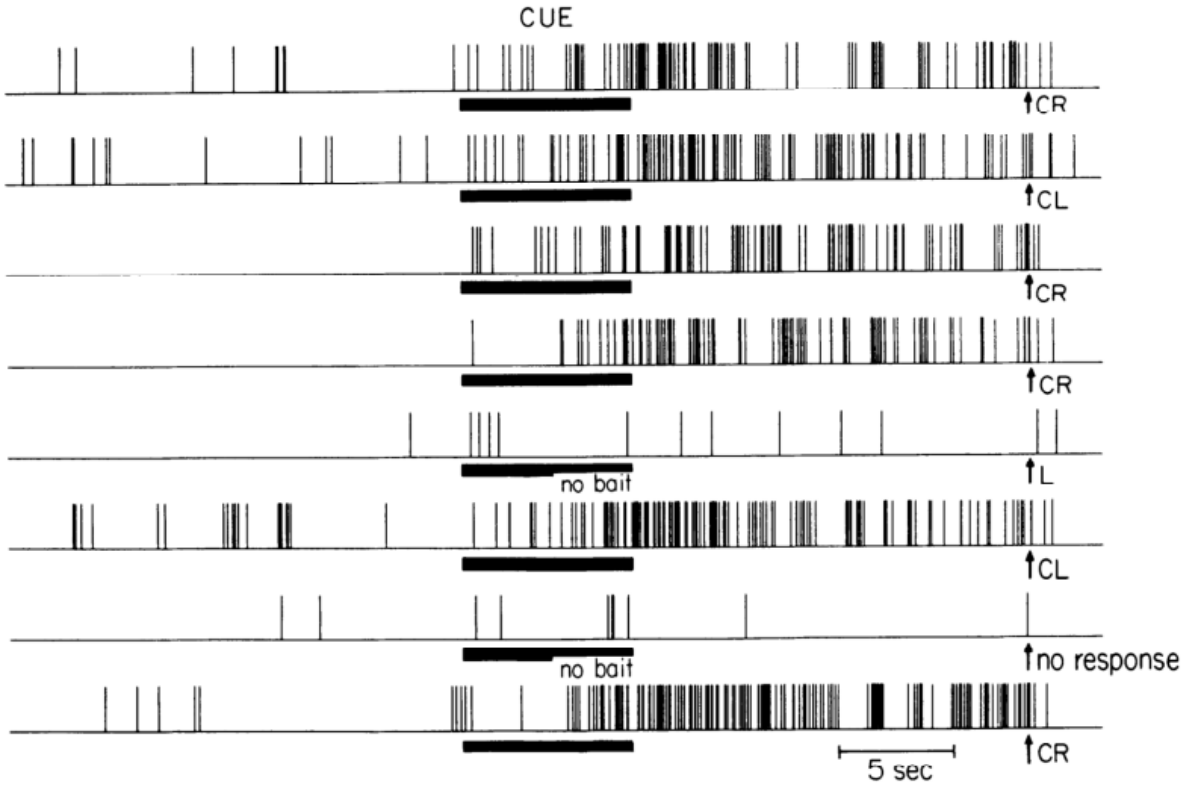
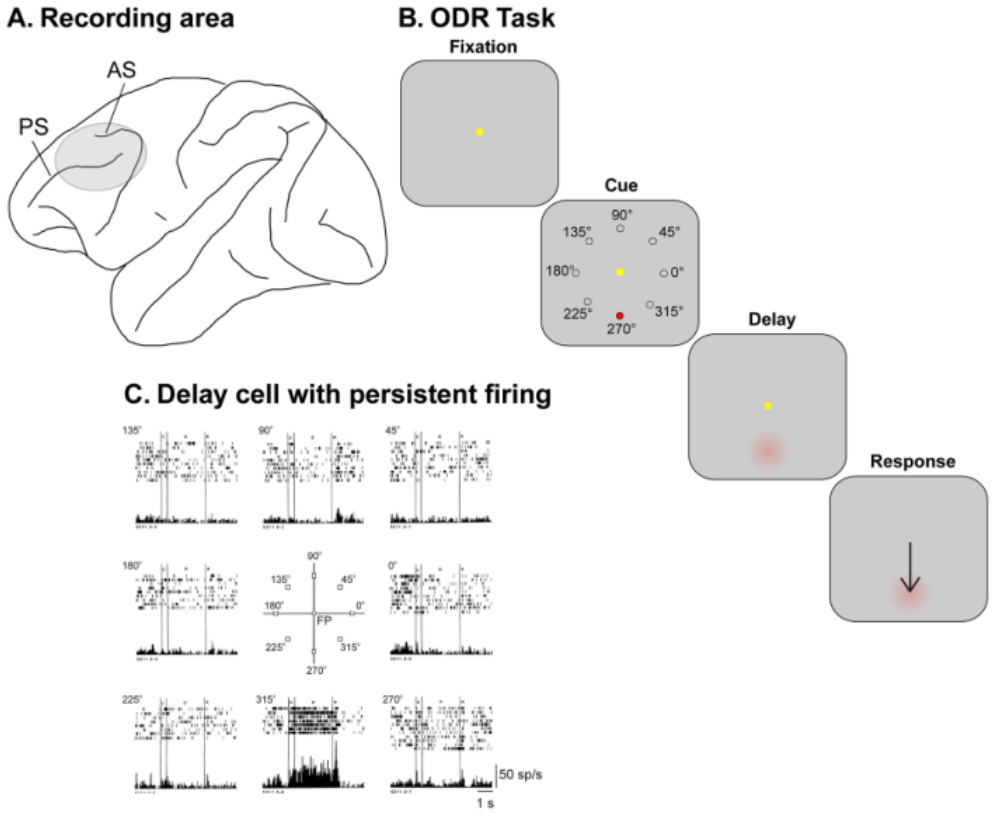


FIG. 6. Unit of type C. Note absence of sustained activation on dry-run trials (fifth and seventh).

Figure 6 of Fuster (1973).

A "modern" version of Fuster's paradigm

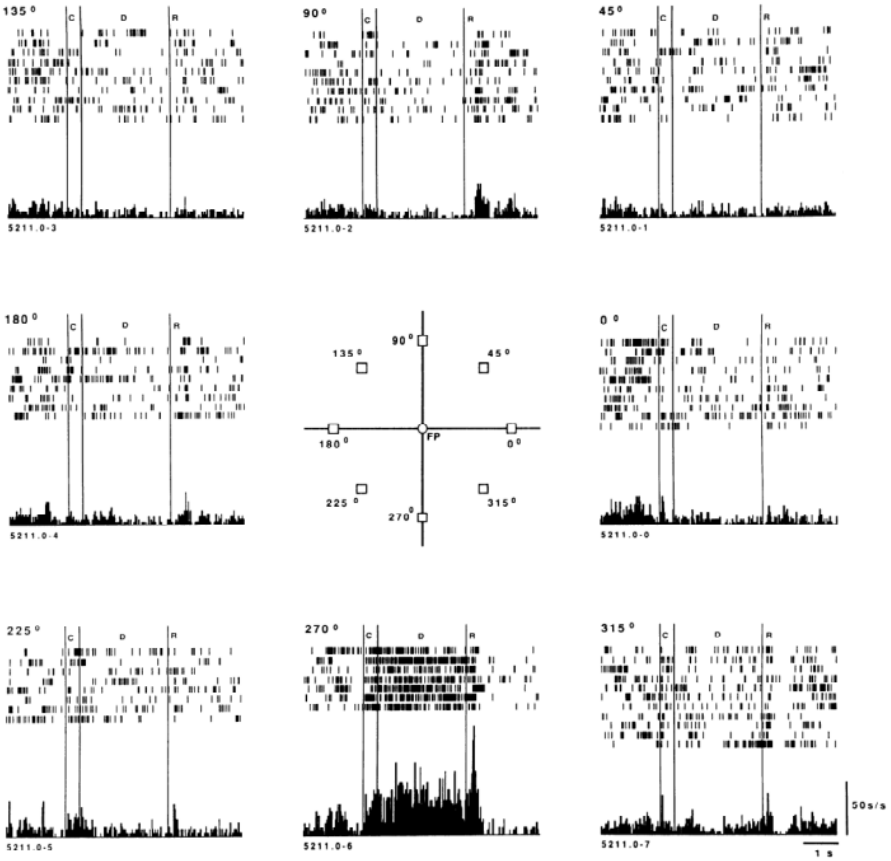


Adaptation of figures from Funahashi et al (1989) by Constantinidis et al (2018).

References:

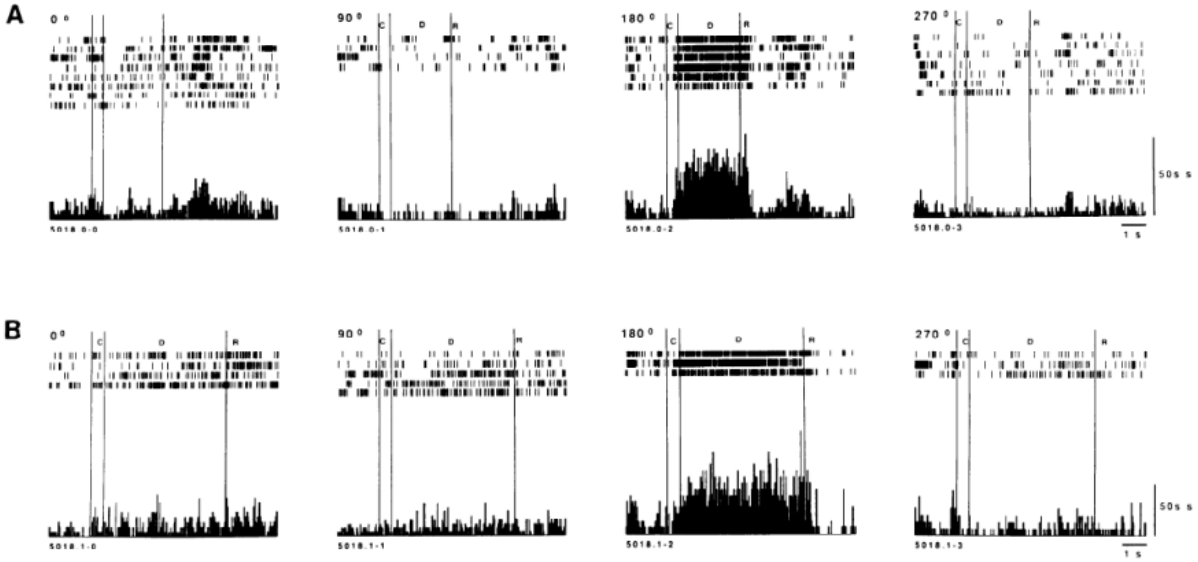
- S. Funahashi, C. J. Bruce, and P. S. Goldman-Rakic (1989) Mnemonic coding of visual space in the monkey's dorsolateral prefrontal cortex . *J. Neurophys.* **61**: 341-349.
- Christos Constantinidis, Shintaro Funahashi, Daeyeol Lee, John D. Murray, Xue-Lian Qi, Min Wang and Amy F.T. Arnsten (2018) Persistent Spiking Activity Underlies Working Memory *Journal of Neuroscience* **38 (32)**: 7020-7028.

A better view of the rasters



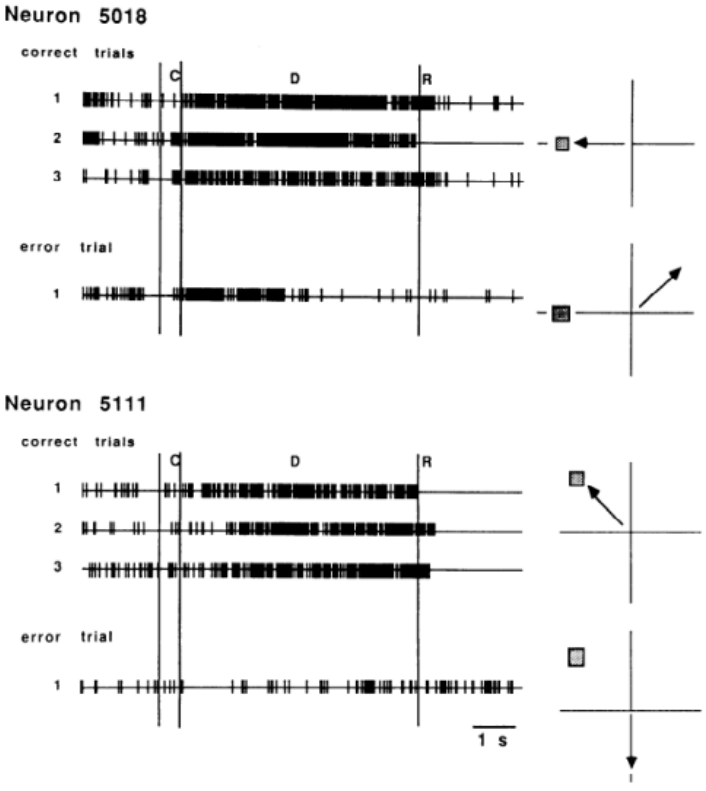
Funahashi et al (1989) Figure 3.

Changing the delay



Funahashi et al (1989) Figure 11.

What happens when mistakes are made?



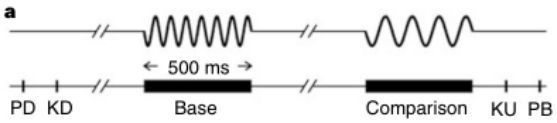
Funahashi et al (1989) Figure 13.

Parametric working memory

Neuronal correlates of parametric working memory in the prefrontal cortex

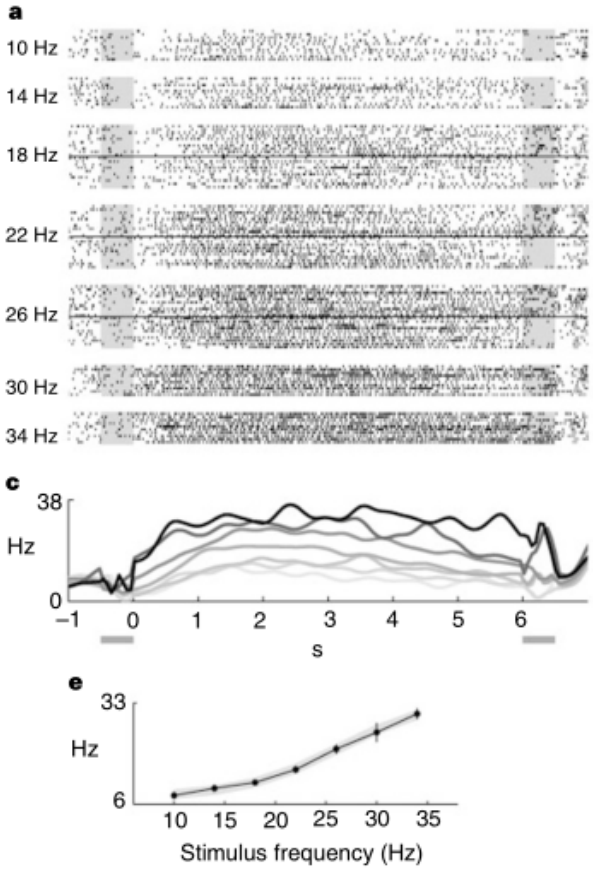
Ranulfo Romo, Carlos D. Brody, Adrián Hernández & Luis Lemus

Instituto de Fisiología Celular, Universidad Nacional Autonóma de México, México D.F. 04510, México



Romo et al (1999) title and figure 1a.

Reference: Romo, R., Brody, C., Hernández, A. et al. Neuronal correlates of parametric working memory in the prefrontal cortex. *Nature* **399**, 470–473 (1999). https://doi.org/10.1038/20939.



Part of Romo et al (1999) figure 2.

First modelling efforts

Synaptic reverberation underlying mnemonic persistent activity

Xiao-Jing Wang

Stimulus-specific persistent neural activity is the neural process underlying active (working) memory. Since its discovery 30 years ago, mnemonic activity has been hypothesized to be sustained by synaptic reverberation in a recurrent circuit. Recently, experimental and modeling work has begun to test the reverberation hypothesis at the cellular level. Moreover, theory has been developed to describe memory storage of an analog stimulus (such as spatial location or eye position), in terms of continuous 'bump attractors' and 'line attractors'. This review summarizes new studies, and discusses insights and predictions from biophysically based models. The stability of a working memory network is recognized as a serious problem; stability can be achieved if reverberation is largely mediated by NMDA receptors at recurrent synapses.

persistent activity to subserve working memory, it must be stimulus-selective, and therefore information-specific. Moreover, it must be able to be turned on and switched off rapidly (≈100 ms) by transient inputs.

For 30 years, persistent activity in the cortex has been documented by numerous unit recordings from behaving monkeys during working memory tasks (Box 1). How does stimulus-selective persistent activity arise in a neural network? Can we explain persistent activity in terms of the biophysics of neurons and synapses, and circuit connectivity?

Reference: Wang XJ. Synaptic reverberation underlying mnemonic persistent activity. *Trends Neurosci.* 2001 Aug;**24(8)**:455-63. doi: 10.1016/s0166-2236(00)01868-3.

Cellular substrate

Heterogeneity in the pyramidal network of the medial prefrontal cortex

Yun Wang[1], Henry Markram[2], Philip H Goodman[3], Thomas K Berger[2], Junying Ma[1] & Patricia S Goldman-Rakic[4,5]

The prefrontal cortex is specially adapted to generate persistent activity that outlasts stimuli and is resistant to distractors, presumed to be the basis of working memory. The pyramidal network that supports this activity is unknown. Multineuron patch-clamp recordings in the ferret medial prefrontal cortex showed a heterogeneity of synapses interconnecting distinct subnetworks of different pyramidal cells. One subnetwork was similar to the pyramidal network commonly found in primary sensory areas, consisting of accommodating pyramidal cells interconnected with depressing synapses. The other subnetwork contained complex pyramidal cells with dual apical dendrites displaying nonaccommodating discharge patterns; these cells were hyper-reciprocally connected with facilitating synapses displaying pronounced synaptic augmentation and post-tetanic potentiation. These cellular, synaptic and network properties could amplify recurrent interactions between pyramidal neurons and support persistent activity in the prefrontal cortex.

Reference: Wang, Y., Markram, H., Goodman, P. H., Berger, T. K., Ma, J., & Goldman-Rakic, P. S. (2006). Heterogeneity in the pyramidal network of the medial prefrontal cortex. *Nature Neuroscience*, **9(4)**, 534–542. doi:10.1038/nn1670.

This is not the whole story!

- NMDA receptors are also involved: Min Wang, Yang Yang, Ching-Jung Wang, Nao J. Gamo, Lu E. Jin, James A. Mazer, John H. Morrison, Xiao-Jing Wang, Amy F.T. Arnsten (2013) NMDA Receptors Subserve Persistent Neuronal Firing during Working Memory in Dorsolateral Prefrontal Cortex. *Neuron*, **77 (4)**: 736-749.
- Dopamine also plays a key role: MIN WANG, SUSHEEL VIJAYRAGHAVAN, PATRICIA S. GOLDMAN-RAKIC (2004) Selective D2 Receptor Actions on the Functional Circuitry of Working Memory. *SCIENCE*, **303**: 853-856

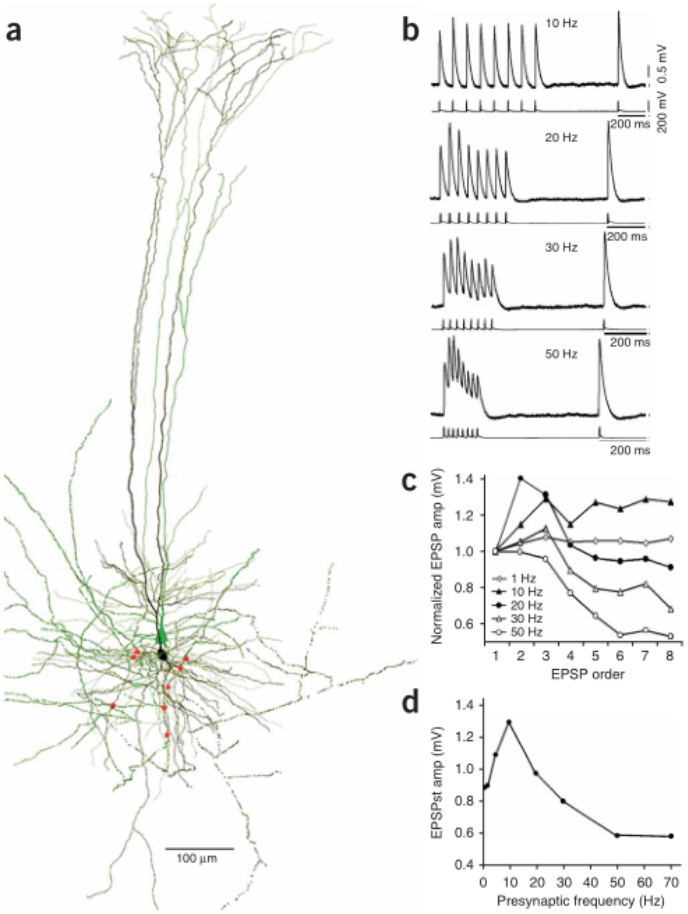


Figure 1 of Wang et al (2006).

Models with short term facilitation

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ScienceDirect

Current Opinion in Neurobiology

Working models of working memory

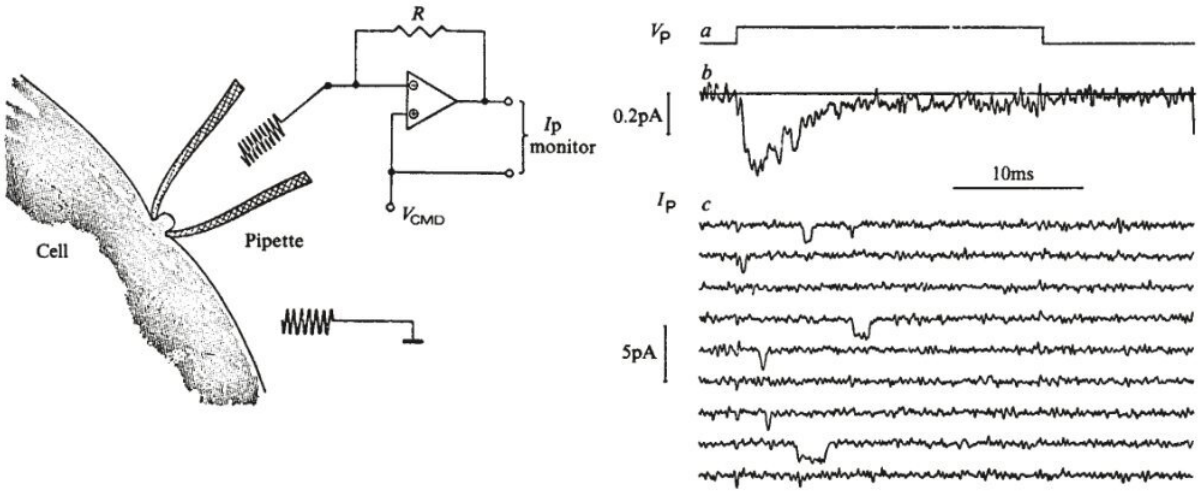
Omri Barak[1] and Misha Tsodyks[2]

Working memory is a system that maintains and manipulates information for several seconds during the planning and execution of many cognitive tasks. Traditionally, it was believed that the neuronal underpinning of working memory is stationary persistent firing of selective neuronal populations. Recent advances introduced new ideas regarding possible mechanisms of working memory, such as short-term synaptic facilitation, precise tuning of recurrent excitation and inhibition, and intrinsic network dynamics. These ideas are motivated by computational considerations and careful analysis of experimental data. Taken together, they may indicate the plethora of different processes underlying working memory in the brain.

activity related to storing a fixed item is not stationary, and there is a large heterogeneity in the firing profiles of different neurons [3,4,5•,6]. From the computational side, the network activity representing a memorized item should exhibit a sufficient degree of stability to ensure memory retainment. This requirement is especially challenging for storing continuous variables, such as orientation or spatial position of a visual cue, because of an inevitable drift along the variable's representation. Furthermore, integrating the various data-driven challenges in a self-consistent manner is often a non-trivial computational problem.

Reference: Omri Barak, Misha Tsodyks (2014) Working models of working memory, *Current Opinion in Neurobiology*, **25**: 20-24.

Membrane conductances (ion channels) generate fluctuations



Figures 1 and 2 of Sigworth and Neher (1980). Reference: Sigworth, F. J., & Neher, E. (1980). Single Na+ channel currents observed in cultured rat muscle cells. *Nature*, **287**: 447-449.

Synapses generate even more fluctuations

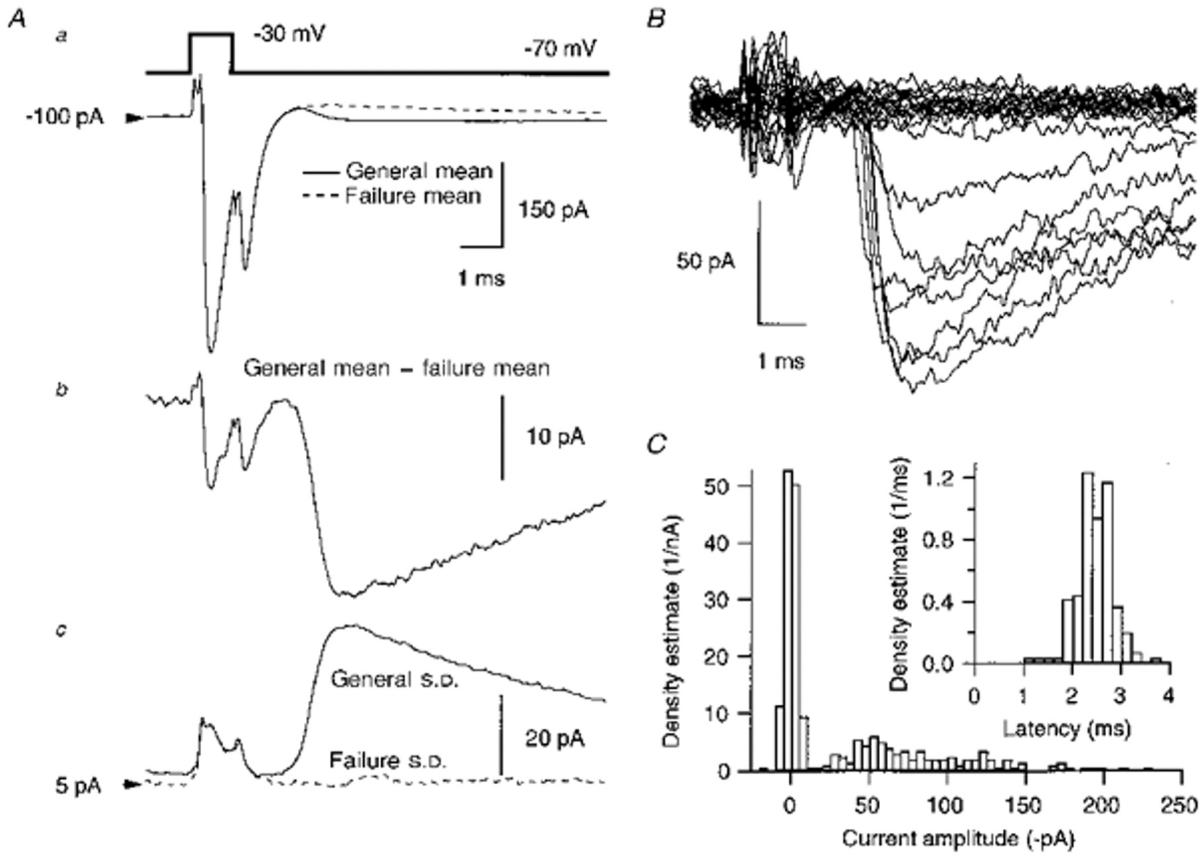


Figure 1 of Pouzat and Marty (1998).

Reference: Pouzat, C., & Marty, A. (1998). Autaptic inhibitory currents recorded from interneurones in rat cerebellar slices. *The Journal of Physiology*, **509(Pt 3)**, 777.

Short Term Plasticity - STP

- STP : change in the *synaptic efficacy*
- acts on timescales which are of the order of hundreds to thousands of milliseconds
- comparable to the timescale of the *spiking activity of the network*
- **3 basic mechanisms of STP** – here, I will only speak about short-term potentiation.

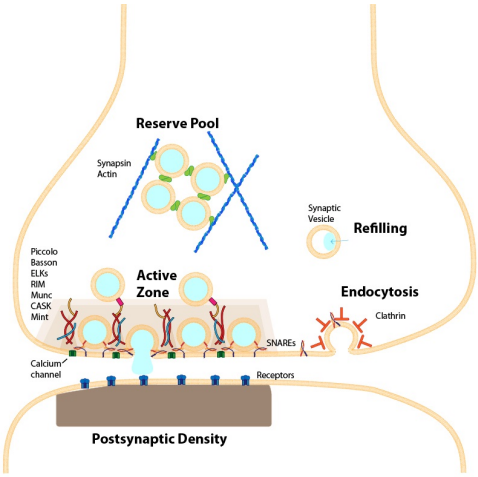


Figure 1: Curtis Neveu, CC BY-SA 3.0.

More on STP

Short-term potentiation due the residual calcium (presynaptic) :

- release of neurotransmitters is triggered by **calcium,** via **calcium channels**.
- after transmitter release : free calcium gets pumped out of the presynaptic terminal.
- but : this takes time.
- So : if a new action potential arrives, it can find a residual calcium concentration that was not pumped out yet, leading to a larger calcium concentration than after the first AP and to a larger transmitter release.

Bibliographic context

- many papers have been devoted to STP since at least the last two decades...
- probably starting with Markram and Tsodyks (1996), followed by Kistler and van Hemmen (1999), Seeholzer et al. (2018) : study of the effect of STP on "working memory"
- mostly : numerical studies of properties of STP within relatively simple phenomenological models
- There is an interesting article *http://www.scholarpedia.org/article/Short-term_synaptic_plasticity* from Scholarpedia
- For the biological aspects : Zucker and Regehr (2002).

2 A model

Our contribution

- We propose a simple mathematical model for short-term potentiation due to the residual calcium
- in which short time memory can be described
- as tendency of the system to keep track of an initial stimulus
- Namely : by staying within a certain **untypical** region of the space of configurations during a short but macroscopic amount of time
- before finally being kicked out of this region and relaxing to the true equilibrium of the process.
- Main technical tool: **a rigorous justification of the passage to a large population limit** and study of the limit model which is described by a $2d-$dynamical system.

The model

- N interacting neurons (N will tend to $+\infty$ later). Purely excitatory case.
- For each neuron i, its membrane potential process $U_t^N(i) \geq 0$ accumulates the stimuli coming from the other neurons. These stimuli are modulated by their current residual calcium values.
- Neuron i spikes at rate $\varphi(U_t^N(i))$, that is,

$$P(\text{ i spikes in }]t, t+h]| \text{ history } \leq t) = \varphi(U_t^N(i))h + o(h).$$

- φ is the spiking rate function of each neuron, Lipschitz, $\varphi(0) = 0$, $\varphi(x) > 0$ for all $x > 0$.
- $Z_t^N(i)$ = counting process counting the spikes of i up to time t.
- Each time i spikes, it gives an additional amount of potential $W_{i\to j}$ to each neuron j in the system.
- $W_{i\to j}$ depends on time, since it depends on the current residual calcium concentration $R_t^N(i)$ at that time :

$$W_{i\to j}(t) = \frac{\alpha}{N} R_{t-}^N(i), \quad \alpha > 0$$

- Continuous leakage of potential continuously at constant rate $\beta > 0$:

$$U_t^N(i) = e^{-\beta t} U_0^N(i) + \frac{\alpha}{N} \sum_{j=1}^{N} \int_{]0,t]} e^{-\beta(t-s)} R_{s-}^N(j) dZ_s^N(j).$$

- The residual calcium concentration of a neuron is increased by 1 at each spike time of the neuron.
- Continuous leakage at constant rate $\lambda > 0$.
- Thus,

$$R_t^N(i) = e^{-\lambda t} R_0^N(i) + \int_{]0,t]} e^{-\lambda(t-s)} dZ_s^N(i),$$

for all $t \geq 0$.

First Remarks

- There is **no reset to a resting potential** after spiking.
- We are only dealing with the purely excitatory case, no inhibition is present in our model. No mathematical problem to add inhibitory synapses...
- Our model can be seen as a system of interacting pairs of coupled Hawkes processes.
- No (direct) interactions in the residual calcium values $R_t^N(i)$: they are just functionals of $Z^N(i)$.

Longtime behavior of the finite system

- Since $\varphi(0) = 0$, the all-zero state is an invariant state of the system.
- Following the ideas of Duarte and Ost (2016), it is straightforward to show :

  Theorem 2.1. *If φ is differentiable in 0, then the system stops spiking almost surely. As a consequence, the unique invariant measure of the process (U_t^N, R_t^N) is given by $\delta_{\mathbf{0},\mathbf{0}}$, where $\mathbf{0} \in \mathbb{R}^{\mathbf{N}}$ denotes the all-zero vector in $\mathbb{R}^N$.*

- Situation changes as $N \to \infty$, that is, number of neurons tends to infinity.

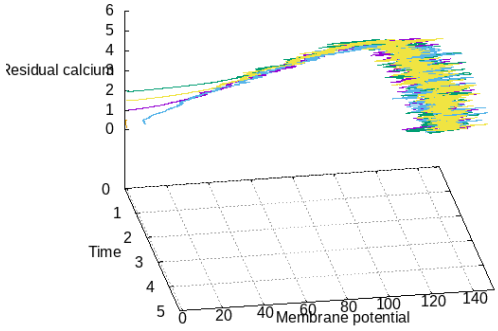


Figure 2: 3D plot of 5 trajectories obtained by simulating a network of 1000 neurons from 5 different initial states.

3 Large population limits

Heuristics : Form of the limit model I

- The membrane potentials of the N neurons evolve according to

$$\begin{aligned} dU_t^N(i) &= -\beta U_t^N(i)dt + \\ &\quad \frac{\alpha}{N}\sum_{j=1}^{N}\int_0^\infty R_{t-}^N(j)1_{\{z\le\varphi(U_{t-}^N(j))\}}\mathbf{M}^j(dt,dz), \end{aligned}$$

where the $\mathbf{M}^j, 1\le j\le N$ are independent Poisson random measures having intensity $dtdz$.

- Limit should be obtained by propagation of chaos : for large N,

$$\frac{\alpha}{N}\sum_{j=1}^{N}\int_0^\infty R_{t-}^N(j)1_{\{z\le\varphi(U_{t-}^N(j))\}}\mathbf{M}^j(dt,dz) \sim$$

$$\mathbb{E}\int_0^\infty \alpha R_{t-}^N(1)1_{\{z\le\varphi(U_{t-}^N(1))\}}\mathbf{M}^1(dt,dz).$$

Heuristics : Form of the limit model II

$$\mathbb{E}\int_0^\infty \alpha R_{t-}^N(1)1_{\{z\le\varphi(U_{t-}^N(1))\}}\mathbf{M}^1(dt,dz)$$

$$= \mathbb{E}\left(\int_0^\infty \alpha R_{t-}^N(1)1_{\{z\le\varphi(U_{t-}^N(1))\}}dz\right)dt = \alpha\mathbb{E}[R_t\varphi(U_t)]dt.$$

So : In the $N\to\infty-$limit, any neuron should behave as an independent copy of

$$\begin{aligned} U_t &= U_0 - \beta\int_0^t U_s ds + \alpha\int_0^t \mathbb{E}[\varphi(U_s)R_s]ds, \\ R_t &= R_0 - \lambda\int_0^t R_s ds + \int_0^t\int_0^\infty 1_{\{z\le\varphi(U_{s-})\}}\mathbf{M}(ds,dz). \end{aligned}$$

Heuristics : Form of the limit model III

$$\begin{aligned} U_t &= U_0 - \beta\int_0^t U_s ds + \alpha\int_0^t \mathbb{E}[\varphi(U_s)R_s]ds, \\ R_t &= R_0 - \lambda\int_0^t R_s ds + \int_0^t\int_0^\infty 1_{\{z\le\varphi(U_{s-})\}}\mathbf{M}(ds,dz). \end{aligned}$$

Remark 3.1. *– Only randomness for the membrane potential process of a typical neuron in the limit is in the initial condition U_0. – If U_0 deterministic, then the spike counting process of a typical neuron in the limit population $t\mapsto\int_0^t\int_0^\infty 1_{\{z\le\varphi(U_{s-})\}}\mathbf{M}(ds,dz)$ is an inhomogeneous Poisson process.*

Heuristics : Form of the limit model IV

Two first questions :

- (Unique) Existence of the limit process?
- **Does the finite system converge to the limit - and how fast?**
- Longtime behavior of the finite and the limit system?
- Answer to the first question : We will have to take care of the product term $\mathbb{E}[\varphi(U_s)R_s]ds$ appearing in the limit dynamics of U : it is non-Lipschitz.
- Answer to the second question : by coupling. See later. The non-Lipschitz term might still cause some troubles...

Well-posedness of the limit process

$dU_t = -\beta U_t dt + \alpha \mathbb{E}[\varphi(U_t)R_t]dt$, $dR_t = -\lambda R_t dt + \int_0^\infty 1_{\{z\le\varphi(U_{t-})\}}\mathbf{M}(dt,dz)$.

First situation : $U_0 = u_0$ deterministic and $\varphi(x) \le C\sqrt{x}$ for all large $x \ge x_0$.

- Then $U_t = u_t$ deterministic, s.t. $\mathbb{E}[\varphi(U_t)R_t] = \varphi(u_t)\mathbb{E}(R_t)$.
  Putting $r_t = \mathbb{E}(R_t)$, (u_t, r_t) solves
  $$du_t = -\beta u_t dt + \alpha\varphi(u_t)r_t dt, dr_t = -\lambda r_t + \varphi(u_t)dt.$$
- Consider $g_t = \sqrt{u_t}$, then for large x,
  $$\dot g_t \le -(\beta/2)g_t dt + C(\alpha/2)r_t dt, \dot r_t \le -\lambda r_t + Cg_t dt.$$
  So : Existence of non-exploding solution whence uniqueness on $[0,T]$ by a priori bounds on r_t and g_t .

Well-posedness of the limit process - BIS

Second situation : U_0 random, but : existence of exponential moments for R_0 and φ bounded and Lipschitz.

- Truncation argument :
  $$|\varphi(U_t)R_t - \varphi(\tilde U_t)\tilde R_t| \le rL_\varphi|U_t - \tilde U_t| + R_t 1_{\{R_t>r\}}\|\varphi\|_\infty + \|\varphi\|_\infty|R_t - \tilde R_t|.$$
- Existence of exponential moments for R_t, uniformly over bounded time intervals.
- For $x_t := \mathbb{E}|\ U_t - \tilde U_t|, y_t = \mathbb{E}|R_t - \tilde R_t|$ and $\alpha_t = x_t + y_t$, clever choice of $r = r_t$ implies
  $$d\alpha_t \le C_T|\ln\tilde\alpha_t|\tilde\alpha_t dt$$
  Osgood's lemma ($x\ln x-$version of Gronwall) implies $\alpha \equiv 0$.

Coupling with the limit system
Still in the above second situation :

- Osgood's lemma + Sznitman coupling (use the same PRM's for the finite and the limit system)
  $$\mathbb{E}\left(|U_t^N(i) - U_t^\infty(i)| + |R_t^N(i) - R_t^\infty(i)|\right) \le C_T N^{-\frac{1}{2}e^{-C_T}}.$$
  (very bad rate in N)
- Can be improved under the condition $\mathbb{E}e^{aR_0\ln R_0} < \infty$ for some $0 < a \le 1$. Then
  $$\mathbb{E}\left(|U_t^N(i) - U_t^\infty(i)| + |R_t^N(i) - R_t^\infty(i)|\right) \le C_T N^{-\frac{1}{2+\varepsilon}}, \quad (1)$$
  for all $N \ge N_0$, $t \le T$.
- Proof relies on the fact that $V(r) = e^{ar\ln r}$ is a Lyapunov function for R_t.

Coupling with the limit system
In the first situation : $U_0^i = u_0$ **deterministic for all** i, φ **bounded and Lipschitz :**

- Then $U_t^N(i) = U^N(i)$ for all i, for all $t \ge 0$, where
  $$dU_t^N = -\beta U_t^N dt + \frac{\alpha}{N}\sum_j R_{t-}^N(j)\int_0^\infty 1_{\{z\le\varphi(U_{t-}^N)\}}M^i(dt,dz).$$
- So we need a control of the deviations of the empirical mean $\frac{1}{N}\sum_{j=1}^N R_t^N(j)$ from the limit expectation.

Theorem 1. *Imposing exponential moments for the initial condition of $R_0^N(i)$, with exp high probability, for any fixed $T > 0$,*
$U^N(i)$ and $\frac{1}{N}\sum_{j=1}^N R_T^N(j)$ stay in a $N^{-1/2+\delta}-$tube around their associated limit quantity during $[0,T]$.

4 Study of the limit process

Study of the limit process in the first framework $U_0 = u_0$

– Let $u_t = \mathbb{E}U_t, r_t = \mathbb{E}R_t$. Then

$$\left\{ \begin{array}{rcl} du_t & = & -\beta u_t dt + \alpha\varphi(u_t) r_t dt \\ dr_t & = & -\lambda r_t dt + \varphi(u_t) dt \end{array} \right\}. \tag{2}$$

– Any stationary solution (u^*, r^*) of (2) must satisfy

$$\lambda r^* = \varphi(u^*) \quad \text{and} \quad \beta u^* = \alpha\varphi(u^*) r^*. \tag{3}$$

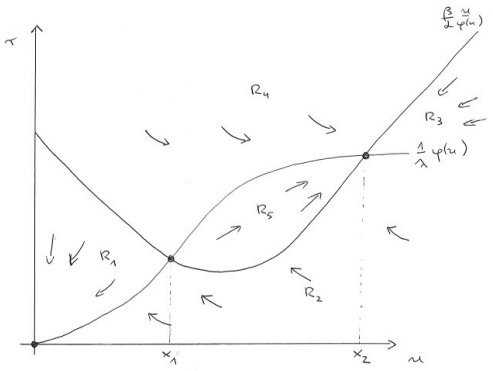
– This implies

$$u^* = \frac{\alpha}{\beta\lambda}\varphi^2(u^*). \tag{4}$$

– Of course, $(0, 0)$ is still an equilibrium.
But for φ sigmoidal, other equilibria might appear.

- $r-$null-cline is given by $\{r = \frac{1}{\lambda}\varphi(u)\}$.
- $u-$null-cline $\{r = (\beta/\alpha)\Phi(u), u > 0\} \cup \{r = u = 0\}$, where $\Phi(u) = u/\varphi(u)$.

Phase portrait :



And for a system that we have simulated :

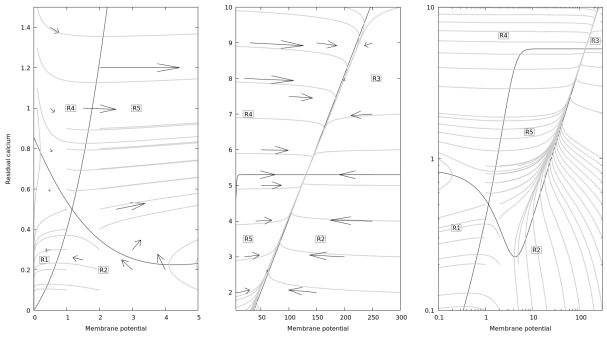


Figure 3: Phase portrait of the limit equation, right figure : on a log-log scale

So : for suitable choices of α, β, λ, a second stable equilibrium (u^{max}, r^{max}) appears - which does not exist for the finite size system.

Theorem 4.1 (Deviations from the limit system)**.** *Take some starting point* (u_0, r_0) *in the domain of attraction of* (u^{max}, r^{max}) *and introduce*

$$t_1 = t_1(\varepsilon) = \inf\{t : (u_t, r_t) \in B_\varepsilon(u^{max}, r^{max})\}.$$

Then for all $N \geq N_1$*, for all* $1 \leq i \leq N$*,*

$$P(|U^N_{t_1}(i) - u^{max}| \geq 2\varepsilon \text{ or } |\frac{1}{N}\sum_{j=1}^{N} R^N_{t_1}(j) - r^{max}| \geq 2\varepsilon) \leq C_{t_1} e^{-c\varepsilon^2\sqrt{N}}.$$

Short term memory

- Suppose we expose the finite (but big) system to some initial stimulus s.t. it is pushed into $B_\varepsilon(u^{max}, r^{max})$.
- Then this stimulus is switched off, and we start observing the system.
- Since the point is attracting and N large, the system is attracted to a small neighbourhood of (u^{max}, r^{max}) and stays in this neighbourhood for a while.
- We interpret this transient behavior as an expression of short term memory.
- In the long run, the system will finally get kicked out of this neighbourhood and start rapidly decaying towards the all-zero state.

Simulations

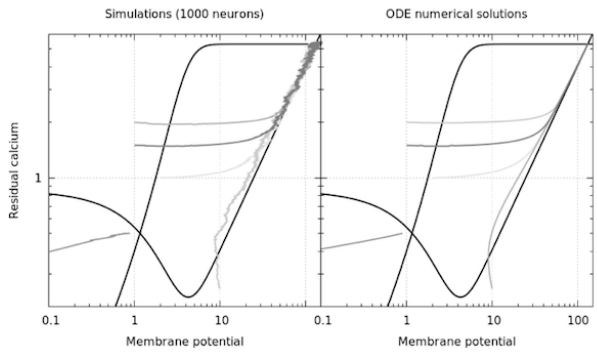
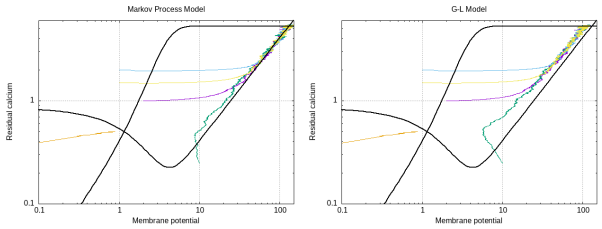


Figure 4: Phase plots on a log-log scale. Left, 5 trajectories (gray lines) obtained by simulating a network of 1000 neurons. On both plots the black curves show the null-cline of u (V shaped) and of r (inverted L shape).

The same behavior should be true for the model with reset (GL)

- We studied model without reset because the limit model is deterministic dynamical system.
- However, simulations show the GL model should behave the same (still needs to be proven).

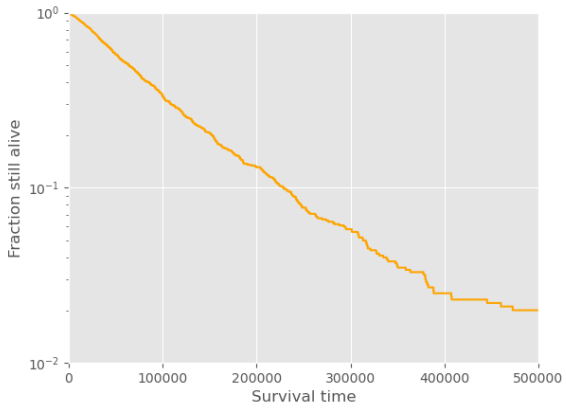


Short term memory and metastability

- All this should be related to **metastability**.
- **Penrose and Lebowitz, Rigorous treatment of metastable states in the van der Waals-Maxwell theory, JSP 1971**
  - A system starting in a metastable state is very likely to stay there for a long time.
  - Once it has left the vicinity of the metastable state, it is very unlikely to return there.
  - This happens after an **unpredictable time** - which is random, and the "unpredictability" is expressed through the fact that it must be *exponentially* distributed.
- Writing τ^N = exit time of $B_\varepsilon(u^{max}, r^{max})$, we believe that we have convergence in law, as $N \to \infty$, of the rescaled exit times

$$\tau^N / \mathbb{E}(\tau^N) \to exp(1).$$

Exponentiality of the last spiking time



(Has only been rigorously proved for a slightly different model including reset, without short term synaptic facilitation : Löcherbach + Monmarché, IHP 2022.)

Empirical survival function obtained from 1000 replicates with $\alpha = 107.78$, $\beta = 50$, $\lambda = 2.16$, $a = 3$ and a **network size of 20**. All simulations start with their membrane potential and residual calcium within 1 % of the asymptotic fixed point values. They run until activity dies (the sum of the *phi* is smaller than 10^{-6}) or until time 500,000 is reached. **A log scale is used for the ordinate**.

Some ideas of the techniques used for our proofs

- We used deviation inequalities for martingales which are of the form

$$M_t^N = \frac{\alpha}{N} \sum_{j=1}^{N} \int_0^t \int_0^\infty R_{s-}^N(j) 1_{\{z \le \varphi(U_{s-}^N(j))\}} [\mathbf{M}^j(ds, dz) - dsdz].$$

- To control these pure-jump martingales having (a priori) **unbounded** jumps, we use a Bernstein-type inequality $\Rightarrow$ see next slide.

Bernstein-type inequality for square integrable martingales with unbounded jumps

$-[M]_t = \sum_{s \le t} (\Delta M_s)^2$, and let $< M >_t$ be its predictable compensator.

– To deal with the big jumps, one puts, for a fixed a,

$$H_t^a := \sum_{s \le t} (\Delta M_s)^2 1_{\{|\Delta M_s| > a\}} + < M >_t .$$

Theorem 4.2 (Dzhaparidze, van Zanten, SPA 2001)**.**

$$P(M_t^* \ge z, H_t^a \le L) \le 2 \exp\left(-\frac{1}{2} \frac{z^2}{L} \psi(\frac{az}{L}) \right), \tag{5}$$

where $\psi(x) = (1 + x/3)^{-1}$ *and* $M_t^* = \sup_{s \le t} |M_s|$.

Some literature

- Duarte, A., Ost, G. A model for neuronal activity in the absence of external stimuli. MPRF 2016
- Kistler, W.M., van Hemmen, L. Short-Term Synaptic Plasticity and Network Behavior. Neural Computation 1999.
- Markram, H., Tsodyks, M. Redistribution of synaptic efficacy between neocortical pyramidal neurons. Nature 1996

Thanks for your attention - our paper has appeared in JSP 2020 and you can find it also on arXiv : https://arxiv.org/abs/1903.01270.

The fully documented simulation codes as well as the `gnuplot` and `Python` analysis scripts used in this presentation can be obtained from our `PlmLab` depository: interacting_neurons_with_stp.